

**The Yukawa Lagrangian Density  
is Inconsistent with the Hamiltonian**

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Abstract:

It is proved that no Hamiltonian exists for the real Klein-Gordon field used in the Yukawa interaction. The experimental side supports this conclusion.

About 70 years ago, the Yukawa interaction was proposed as a quantum mechanical interpretation of the nuclear force (see [1], p. 78). This interaction is derived from the Lagrangian density of a system of a Dirac field and a Klein-Gordon (KG) field (see [2], p. 79)

$$\mathcal{L}_Y = \mathcal{L}_D + \mathcal{L}_{KG} - g\phi\bar{\psi}\psi. \quad (1)$$

Here the first term on the right hand side represents the Lagrangian density of a free Dirac field (see [2], p. 43)

$$\mathcal{L}_D = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \quad (2)$$

and the second term represents the Lagrangian density of a free KG field (see [2], p. 16)

$$\mathcal{L}_{KG} = \frac{1}{2}(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2). \quad (3)$$

The last term of (1) represents the interaction. Since the Hamiltonian is a Hermitian operator, the KG function  $\phi$  used here is real.

In this work, Greek indices run from 0 to 3 and Latin indices run from 1 to 3. The Lorentz metric is diagonal and its entries are (1,-1,-1,-1). Units where  $\hbar = c = 1$  are used. The symbol  $_{,\mu}$  denotes the partial differentiation with respect to  $x^\mu$ .

Difficulties concerning the KG Lagrangian density of a complex KG function have been pointed out recently. Thus, it is proved that a KG particle cannot interact with the electromagnetic fields: an application of the linear interaction  $j^\mu A_\mu$ , where the KG 4-current  $j^\mu$  is independent of the external 4-potential  $A_\mu$ , fails [3]; if the quadratic expression  $(p^\mu - eA^\mu)(p_\mu - eA_\mu)$  is used then the inner product of the Hilbert space of the KG wave function  $\phi$  is destroyed. In addition to that, there is no covariant differential operator representing the Hamiltonian of a complex KG particle [4].

Another difficulty is the inconsistency of the 4-force derived from the Yukawa potential

$$u(r) = -g^2 e^{-mr} / r \quad (4)$$

with the relativistic requirement where the 4-acceleration must be orthogonal to the 4-velocity

$$a^\mu v_\mu = 0. \quad (5)$$

This requirement is satisfied by the electromagnetic interaction, where the Lorentz force is

$$ma^\mu = eF^{\mu\nu}v_\nu. \quad (6)$$

Here the electromagnetic field tensor is antisymmetric  $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$  and this property satisfies (5)

$$a^\mu v_\mu = \frac{e}{m} F^{\mu\nu} v_\nu v_\mu = 0. \quad (7)$$

On the other hand, the scalar function  $\phi$  cannot yield an antisymmetric tensor. Therefore, the force found in the classical limit of the Yukawa interaction is inconsistent with special relativity.

The purpose of the present work is to prove that the Lagrangian density (1) of the real KG field  $\phi$  is inconsistent with the fundamental quantum mechanical equation

$$i\frac{\partial\phi}{\partial t} = H\phi. \quad (8)$$

This task extends the validity range of the proof of [4] where the complex KG field is discussed.

The Euler-Lagrange equations of a given Lagrangian density are obtained from the following general expression (see [2], p. 16)

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi}{\partial x^\mu}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \quad (9)$$

Applying (9) to the KG function  $\phi$  of (1), one obtains an *inhomogeneous* KG equation

$$(\square + m^2)\phi = g\bar{\psi}\psi. \quad (10)$$

The following argument proves that the Euler-Lagrange equation (10) obtained from the Yukawa Lagrangian density (1) is inconsistent with the existence of a Hamiltonian. Indeed, the term  $\partial^2/\partial t^2$  of (10) and the independence of its right hand side on the KG wave function  $\phi$ , prove that it is a *second order inhomogeneous* partial differential equation. On the other hand, the Hamiltonian equation (8) is a *first order homogeneous* equation. Now, assume that at a certain instant  $t_0$ , a solution  $\phi_0$  of (8) solves (10) too. Using the fact that (10) is a second order differential equation, one finds that its first derivative with respect to the time is a free parameter. This degree of freedom proves that an infinite number of different solutions of (10) agree with the *single* solution  $\phi_0$  of (8) at  $t_0$ . Thus, for  $t > t_0$ , just one solution of (10) agrees with the solution of the Hamiltonian (8) and all other solutions differ from it.

Moreover, if  $\phi_0$  solves the *homogeneous* equation (8), then  $c\phi_0$ , where  $c$  is a constant, solves it too. Therefore, at  $t_0$ , an infinite number of solutions that solve the Hamiltonian equation (8) correspond to every solution of the Euler-Lagrange equation (10) obtained from the Yukawa Lagrangian density.

Either of these results prove that the Yukawa Lagrangian density (1) is inconsistent with the existence of a Hamiltonian. It is interesting to note that the Dirac Hamiltonian agrees perfectly with the Euler-Lagrange equation obtained from the Dirac Lagrangian density (2) (see [4], p. 32).

Another problem of the Yukawa Lagrangian density (1) is that its wave function  $\phi$  is real. Hence, the real Yukawa function  $\phi$  cannot be an energy

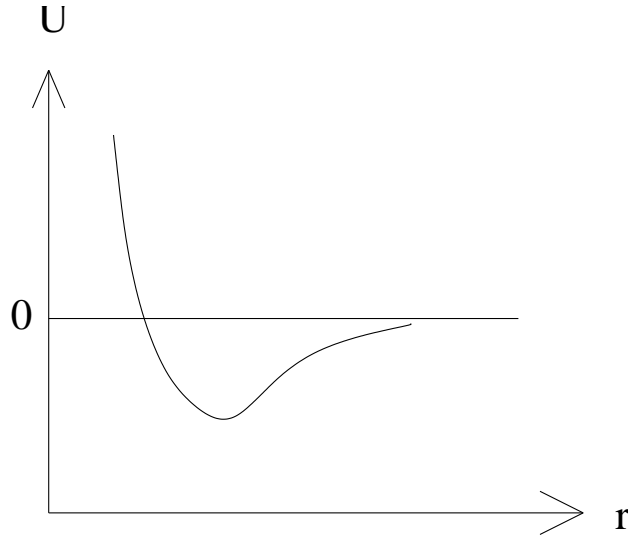


Figure 1: A qualitative description of the nucleon-nucleon phenomenological potential as a function of the distance between the nucleons' centers.

eigenfunction (namely, an eigenfunction of the operator  $i\partial/\partial t$ ), because an energy eigenfunction has a complex factor  $e^{-i\omega t}$ . Therefore, the Yukawa particle cannot be an isolated free particle. This result provides a proof showing that  $\pi^0$  is not a Yukawa particle. Indeed, the lifetime of  $\pi^0$  is about  $10^{-16}$  seconds (see [5], p. 500). Thus, having a relativistic velocity, its path is more than  $10^7$  fermi. This length is much larger than the nucleon's radius which is about 1.2 fermi. Hence,  $\pi^0$  is a free particle for the most of its lifetime, contrary to the above mentioned restriction on a Yukawa particle.

Turning to the experimental side, it is not surprising to find that Nature is unkind to the Yukawa theory. Thus, the Klein-Gordon field function  $\phi(x^\mu)$  depends on a *single* set of space-time coordinates. Hence, like the Dirac field  $\psi(x^\mu)$ , it describes a structureless pointlike particle. Now, unlike Dirac particles (electrons, muons, quarks etc.), the existence of pointlike KG particles

has not been established. In particular, it is now recognized that  $\pi$  mesons, which are regarded as the primary example of a KG particle, are made of a quarks and an antiquark. Hence,  $\pi$  mesons are not pointlike particles. Experimental data confirms this conclusion (see [5], p. 499).

The actual nuclear potential is inconsistent with the Yukawa formula (4). Indeed, the nuclear potential is characterized by a hard (repulsive) core and at its outer side there is a rapidly decreasing attractive force. Its general form is described in fig. 1 (see [1], p. 97).

Thus, the figure proves that the actual nuclear potential and its derivative with respect to  $r$  change sign. This is certainly inconsistent with the Yukawa formula (4). Indeed, neither the Yukawa potential nor its derivative change sign.

These arguments prove that the experimental side and the theoretical analysis carried out above, do not support the validity of the Yukawa theory.

References:

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- [1] S. S. M. Wong, *Introductory Nuclear Physics* (Wiley, New York, 1998).  
2nd edition.
- [2] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, Mass., 1995).
- [3] E. Comay, *Apeiron*, **11**, No. 3, 1 (2004).
- [4] E. Comay, *Apeiron* **12**, no. 1, 27 (2005).
- [5] S. Eidelman et al. (Particle Data Group), *Phys. Lett.* **B592**, 1 (2004).